Probability-fidelity tradeoffs for targeted quantum operations

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We present probability-fidelity tradeoffs for a varying quantum operation with fixed input–output states and for a varying inversion of a fixed quantum operation. © 2009 Elsevier B.V. All rights reserved.

1. Introduction

Since the seminal work in Ref. [1], many information/disturbance tradeoffs have been derived in a wide range of frameworks [2–11]. Despite this variety, all tradeoffs were based on figures of merit defined as average over some ensemble, e.g. the uniform ensemble of all transformations.

In this Letter, following the suggestions contained in Ref. [12], we study the behavior of a single quantum operation in some simple cases, along the following lines. After reviewing the probability of transforming a pair of pure states to another given pair [13], we extend it to mixed target states, and then we provide a tradeoff between the probability and the fidelity of such a transformation. Finally, we present the probability-fidelity tradeoff in the inversion of an atomic (i.e. single-Kraus) quantum operation.

2. State transformations

We are given an ensemble \( \mathcal{E} = \{q_\pm, \ket{\psi_\pm}, \bra{\psi_\pm}\} \) of two pure states \( \ket{\psi_\pm} \) with equal a priori probabilities \( q_\pm = 1/2 \), and a pair of (generally mixed) target states \( \rho_\pm \). We want to find a quantum operation which realizes the transformation

\[
\ket{\psi_\pm} \rightarrow \rho_\pm
\]

(1)

maximizing the mean probability of success over the ensemble.

For pure final states \( \rho_\pm = \ket{\phi_\pm} \bra{\phi_\pm} \) the problem has been solved in Ref. [13]:

Proposition 1. The maximum mean probability is

\[
p = \min \left\{ \frac{1 - |\langle \psi_+ | \psi_- \rangle|}{1 - |\langle \phi_+ | \phi_- \rangle|}, 1 \right\}.
\]

(2)

Moreover, this probability is achieved with a balanced transformation, i.e. a transformation occurring with equal probability on both initial states.

Indeed, the above proposition can be extended also to final mixed states.

Proposition 2. For generally mixed final states \( \rho_\pm \) the maximum mean probability is

\[
p = \min \left\{ \frac{1 - |\langle \psi_+ | \psi_- \rangle|}{1 - F(\rho_+, \rho_-)}, 1 \right\},
\]

(3)

where \( F(\rho, \sigma) := \text{Tr} \sqrt{\rho \sigma} \sqrt{\rho} \) is the Uhlmann fidelity [14]. Moreover, the probability is achieved with a balanced transformation.

Proof. Suppose we have a quantum operation \( \mathcal{E} \) realizing the transformation

\[
\ket{\psi_\pm} \rightarrow \rho_\pm
\]

(4)

with certain probabilities \( p_\pm \). Using the Ozawa dilation theorem [15] for quantum instruments we can realize the quantum operation in the following way

\[
\mathcal{E}(\rho) = \text{Tr}_2 \left[ (I \otimes P) U(\rho \otimes |0\rangle \langle 0|) U^\dagger (I \otimes P) \right],
\]

(5)

where \( |0\rangle \) is any pure state of an ancillary system, \( U \) is a unitary system-ancilla interaction, \( P \) is an orthogonal projector, and we
take the trace on the ancilla. Since unitaries and projectors cannot turn a pure state into a mixed one, the quantum operation $\mathcal{E}$, applied to our initial states $|\psi_\pm\rangle$, will have the form

$$\mathcal{E}(|\psi_\pm\rangle\langle\psi_\pm|) = p_\pm \text{Tr}_2(|\Phi_\pm\rangle\langle\Phi_\pm|),$$

where $|\Phi_\pm\rangle$ are joint ancilla-system states and $p_\pm$ are the success probabilities. Note that $|\Phi_\pm\rangle$ are actually purifications of the final states $\rho_\pm$.

In this way we proved that every state transformation $|\psi_\pm\rangle \rightarrow \rho_\pm$ can be realized with a transformation between pure states $|\psi_\pm\rangle \rightarrow |\Phi_\pm\rangle$ followed by a partial trace. Thus, in order to maximize the probability of $|\psi_\pm\rangle \rightarrow \rho_\pm$, it is not restrictive to search only among those transformations which take $|\psi_\pm\rangle$ into purifications of the final states $\rho_\pm$.

From Uhlmann's theorem [14] we have that

$$|\langle \Phi_+ | \Phi_- \rangle| \leq F(\rho_+, \rho_-),$$

for all the purifications of $\rho_\pm$, and thus

$$1 - |\langle \Phi_+ | \Phi_- \rangle|^2 = 1 - F(\rho_+, \rho_-).$$

From the previous proposition we already know that the maximum probability for $|\psi_\pm\rangle \rightarrow |\Phi_\pm\rangle$ is given by Eq. (2) and thus the upper bound holds

$$p \leq \min \left\{ \frac{1 - |\langle \psi_+ | \psi_- \rangle|^2}{1 - F(\rho_+, \rho_-)}, 1 \right\}.$$  

This bound can be achieved by choosing the purifications $|\Phi_\pm\rangle$ which give the equality in Eq. (7). The transformation is balanced by the previous proposition.

3. Probability/fidelity tradeoff

Let us consider now the transformation

$$|\psi_\pm\rangle \rightarrow |\varphi_\pm\rangle, \quad |\varphi_+ | \varphi_- \rangle | \leq |\langle \psi_+ | \psi_- \rangle|.$$  

By Proposition 1 we know that it can be realized exactly only probabilistically. But if we allow also approximate transformations, realized by quantum operations which transform $|\psi_\pm\rangle$ into states $\rho_\pm$ close to $|\varphi_\pm\rangle$, we have

$$|\psi_\pm\rangle \rightarrow \rho_\pm = \mathcal{E}(|\psi_\pm\rangle\langle\psi_\pm|)/p_\pm, \quad p_\pm = \text{Tr}(\mathcal{E}(|\psi_\pm\rangle\langle\psi_\pm|)),$$

we may be able to implement the transformation with greater probability, or even deterministically.

In general, there are two figures of merit characterizing the transformation: (1) the probability of success, (2) the fidelity between the target states and the states actually obtained. Intuitively, the more we try to tilt the pair $|\psi_\pm\rangle$ towards the target states, the less the transformation is likely to happen.

The figures of merit are defined as follows

$$p = \min\{p_+, p_-, F(|\varphi_\pm\rangle\langle\varphi_\pm|, \rho_+), F(|\varphi_\pm\rangle\langle\varphi_\pm|, \rho_-)\},$$

where $p$ is the minimum probability and $F$ is the minimum fidelity over the two states (a worst-case criterion). Each transformation is characterized by a pair $(p, F)$, the set of all transformations thus being in correspondence with a subset of $[0,1] \times [0,1]$. Our task is to determine the frontier of this permitted subset, thus finding the transformations maximizing both figures of merit.

We can restrict our attention to approximated target states $\rho_\pm$ having the same two-dimensional support, equal to the linear span of the target states $|\psi_\pm\rangle$. In fact, exploiting the Kraus representation for $\mathcal{E}$ [16] (with Kraus operators $K_j$) and defining the unnormalized states $|\beta_\pm\rangle := K_j |\psi_\pm\rangle$, we have

$$\rho_\pm = \frac{1}{p_\pm} \sum_j |\beta_\pm\rangle \langle \beta_\pm|.$$

We note that we can apply unitary operators $U_j$ after the Kraus operators $K_j$ without altering the probabilities, obtaining new states $\rho_\pm'$

$$\rho_\pm' = \frac{1}{p_\pm} \sum_j U_j |\beta_\pm\rangle \langle \beta_\pm| U_j^\dagger,$$

whose fidelity with the target states is

$$F(|\psi_\pm\rangle\langle\psi_\pm|, \rho_\pm') = \frac{1}{p_\pm} \sum_j |\langle \psi_\pm | U_j \rho_\pm' U_j \dagger \rangle|^2.$$

Thus, in order to have $F(|\psi_\pm\rangle\langle\psi_\pm|, \rho_\pm') \geq F(|\psi_\pm\rangle\langle\psi_\pm|, \rho_\pm)$ for $\rho_\pm'$ supported on the span of $|\psi_\pm\rangle$, we only need to show that, given a pair of vectors $|\beta_\pm\rangle$, there is always a unitary transformation $U$ moving $|\beta_\pm\rangle$ in the span of $|\psi_\pm\rangle$ such that

$$|\langle \psi_\pm | U \beta_\pm \rangle|^2 \geq |\langle \psi_\pm | \beta_\pm \rangle|^2.$$  

The operator $U$ can be constructed in the following way.

Let us consider the component of $|\beta_\pm\rangle$ orthogonal to $\text{Span}(\varphi_\pm, |\beta_-\rangle)$. We rotate it into the one-dimensional subspace of $\text{Span}(\varphi_\pm, |\beta_-\rangle)$ orthogonal to $\text{Span}(\varphi_\pm, |\beta_-\rangle)$. In this way, we have moved the four vectors in a three-dimensional space without changing the relevant scalar products $|\langle \psi_\pm | \beta_\pm \rangle|^2$. The intersection $V = \text{Span}(\varphi_\pm, |\beta_-\rangle) \cap \text{Span}(\varphi_\pm, |\beta_-\rangle)$ is one-dimensional, thus we can rotate the components of $|\beta_\pm\rangle$ orthogonal to $V$ into the one-dimensional subspace of $\text{Span}(\varphi_\pm, |\beta_-\rangle)$ orthogonal to $V$. This rotation leaves all vectors in a two-dimensional space and increases the modulus of the scalar products $|\langle \psi_\pm | \beta_\pm \rangle|^2$.

In the following we will then restrict to the span of $|\psi_\pm\rangle$, and it is convenient to use the Bloch representation of states of bidi-}

sensional systems

$$\rho = (I + r \cdot \sigma)/2,$$

where the Bloch vector $r = (x, y, z) \in \mathbb{R}^3$ denotes a point in the unit ball $|r| \leq 1$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. In the Bloch representation the fidelity between the states $\rho$ and $\sigma$ (with Bloch vectors $r_\rho$ and $r_\sigma$) becomes [17]

$$F(\rho, \sigma) = \frac{(1 + r_\rho \cdot r_\sigma + \sqrt{(1 - |r_\rho|^2)(1 - |r_\sigma|^2)})^\frac{3}{2}}{\sqrt{2}},$$

which, when one of the two states is pure simplifies as follows

$$F(\rho, \sigma) = \sqrt{1 + r_\rho \cdot r_\sigma}.$$  

The angle between vectors $r_\rho$, $r_\sigma$ and the angle between vectors $r_\rho$, $r_\sigma$ are both minimized for the pair $r_\rho$, and with the same symmetry axis. This relative position of the couples of vectors can be achieved by a rotation of the couple $r_\rho$ in the Bloch sphere, corresponding to a unitary transformation which leaves the probabilities $p_\pm$ invariant. Now, for each operation $\mathcal{E}$ realizing a certain transformation

$$\mathcal{E}(|\psi_\pm\rangle\langle\psi_\pm|) = p_\pm \rho_\pm,$$

where $p_\pm$ are coplanar with $|\psi_\pm\rangle$, we can construct an operation $\mathcal{E}'$ acting in the following way

$$\mathcal{E}'(|\psi_\pm\rangle\langle\psi_\pm|) = \frac{1}{2} (p_\pm \rho_\pm + p_+ \sigma_x \rho_\pm \sigma_2),$$

where $\rho_\pm$ are coplanar with $|\psi_\pm\rangle$.
where we have chosen the basis of the representation such that \( \sigma_z \) is the \( \pi \)-rotation around the symmetry axis of the pair \(|\varphi_\pm\rangle\), i.e. \( \sigma_z|\varphi_+\rangle = |\varphi_+\rangle \). The second term in the r.h.s. is simply the “mirror image” of \( \mathcal{E} \). This new quantum operation is symmetric since \( \sigma_z\mathcal{E}(|\varphi_\pm\rangle|\varphi_\mp\rangle) = \mathcal{E}|\varphi_\pm\rangle|\varphi_\mp\rangle = \mathcal{E}|\varphi_\pm\rangle|\varphi_\mp\rangle |\varphi_\mp\rangle |\varphi_\pm\rangle \) and behaves better than the original one w.r.t. both figures of merit in Eqs. (12) and (13), since

\[
\text{Tr}(\mathcal{E}'(|\psi_\pm\rangle|\psi_\pm\rangle)) = \frac{1}{2}(p_+ + p_-) \geq \min\{p_+, p_-, \}
\]

This clearly shows that the optimal states maximizing probability (24) are those minimizing \( \beta \). The pair \(|\xi_\pm\rangle\) satisfies this request, whence it is the most probable.

The remaining part of the optimal tradeoff curve can now be completed quite easily: we only need to sweep the pure states in the arc between \(|\varphi_\pm\rangle\) and \(|\varphi_\mp\rangle\) to obtain the points in the \((p, F)\)-plane connecting \((1, f_0)\) and \((p_0, 1)\), where \(p_0 = \left(1 - \langle|\varphi_\pm\rangle|\varphi_\mp\rangle\right) / \left(1 - \langle|\varphi_\pm\rangle|\varphi_\mp\rangle\right)\). After a little trigonometry, we obtain the explicit expression for this part of the curve

\[
F(p) = \cos\left[\arccos\left(|\varphi_\pm|\varphi_\mp\rangle\right) - \arccos\left(1 - \frac{1 - \langle|\varphi_\pm|\varphi_\mp\rangle\rangle}{p}\right)\right].
\]

In Figs. 2–3 we plot these curves for different values of the fidelities \(|\langle|\psi_\pm|\varphi_\pm\rangle\rangle|\) and \(|\langle|\psi_\mp|\varphi_\mp\rangle\rangle|\).

4. Tradeoff for the inversion of a quantum operation

Suppose we want to know whether a given quantum operation \( \mathcal{E} \) can be inverted deterministically on some subspace \( \mathcal{L} \subseteq \mathcal{H} \), in other words whether there is a quantum channel \( \mathcal{R} \) such that

\[
\rho \rightarrow \rho' = \frac{\mathcal{E}(\rho)}{\text{Tr}(\mathcal{E}(\rho))} \rightarrow \mathcal{R}(\rho') = \rho
\]

for every \( \rho \) with \( \text{supp}(\rho) \subseteq \mathcal{L} \). Necessary and sufficient conditions for this inversion have been proved in Ref. [18], while in Ref. [19] an equivalent condition based on information-theoretical quantities such as entropy and coherent information is provided. If the quantum operation cannot be inverted by a channel, or the inversion is not required to be perfect, it is still possible to achieve an approximate inversion which brings \( \rho' \) close to \( \rho \). Such “closeness” has been quantified in Ref. [20], whenever \( \mathcal{E} \) is a channel, and in Ref. [21] for general quantum operations.

In the present Letter we explore the possibility of probabilistic inversions, including exact inversions as a particular case. In the following we will focus on a two-dimensional system undergoing an atomic quantum operation

\[
\mathcal{E}(\rho) = M\rho M^\dagger,
\]

where \( M \) is a contraction, i.e. satisfying \(|M| \leq 1\). Using the polar decomposition \( M = UP \) with unitary \( U \) and \( P \geq 0 \), w.l.o.g. we can take \( M = M_\beta = M^\circ \) with the following matrix representation

\[
M_\beta = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}
\]

where \( \beta, 0 \leq \beta \leq 1, \) is the smallest singular value. The largest singular value can be fixed at 1 up to an overall probability rescaling.
independent of the state (we assume that the quantum operation has happened).

We will consider two case studies with a given set \(D\) of initial states, and a given set \(Q\) of quantum operations inverting \(M_\beta\) approximately. After the transformation on the state \(\rho \in D\)

\[
\rho' = \frac{M_\beta \rho M_\beta}{\text{Tr}(\rho M_\beta^2)},
\]

a following “inverting” quantum operation \(R \in Q\) leaves the system in the state

\[
\rho'' = \frac{R(\rho')}{\text{Tr}(R(\rho'))}.
\]

The quality of the inversion is assessed by two figures of merit:

(i) the probability of success

\[ p(R; \rho) = \text{Tr}(R(\rho')) , \]

(ii) the fidelity with the initial state

\[ f(R; \rho) = F(\rho, \rho''), \]

In order to keep the probability of success above some threshold \(\bar{p}\) we consider only the subset \(Q_{\bar{p}} \subseteq Q\) whose elements \(R\) satisfy the constraint:

\[ p(R; \rho) \geq \bar{p}, \quad \forall \rho \in D . \]

\[ (34) \]

In a worst-case criterion we have to choose the inversion \(\bar{R} \in Q_{\bar{p}}\) maximizing the minimum fidelity over \(D\)

\[ \bar{R} = \arg \max_{R \in Q_{\bar{p}}} \min_{\rho \in D} f(R; \rho) . \]

\[ (35) \]

This gives the point \((\bar{p}, \bar{F})\), with \(\bar{F} = \min_{\rho \in D} f(\bar{R}; \rho)\), in the \((p, F)\) plane. The tradeoff curve is obtained varying \(\bar{p}\) in the interval \([0, 1]\). In this way we obtain a curve \(F = F(p)\) giving the minimum fidelity over \(D\) achievable with probability of success at least \(p\).

4.1. Semiclassical case

The set of input states \(D = \{\rho_x\}\) consists of all density operators jointly diagonal with the contraction

\[ \rho_x = \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix}, \quad 0 \leq x \leq 1, \]

\[ (36) \]

while the set of possible inversions \(Q = \{N_\gamma\}\) consists of the diagonal contractions

\[ N_\gamma = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta \leq \gamma \leq 1. \]

\[ (37) \]

The unit-fidelity case is the matrix inverse (rescaled in order to keep it a contraction) \(N_\beta = M_\beta^{-1} / \|M_\beta^{-1}\|^{1/2}\).

The states \(\rho'_x\) and \(\rho''_x\) are easily computed

\[ \rho'_x = \frac{1}{x + \beta^2(1-x)} \begin{pmatrix} x & 0 \\ 0 & \beta^2(1-x) \end{pmatrix}, \]

\[ (38) \]

\[ \rho''_x = \frac{1}{y^2 x + \beta^2(1-x)} \begin{pmatrix} y^2 x & 0 \\ 0 & \beta^2(1-x) \end{pmatrix}, \]

\[ (39) \]

and so are the probability and the fidelity

\[ p(N_\gamma; \rho_x) = \frac{y^2 x + \beta^2(1-x)}{\sqrt{y^2 x + \beta^2(1-x)}}, \]

\[ (40) \]

\[ f(N_\gamma; \rho_x) = \frac{y^2 x + \beta(1-x)}{\sqrt{y^2 x + \beta^2(1-x)}}. \]

\[ (41) \]

By inspection of these expressions one can see that the set \(Q_{\bar{p}}\) consists of all states \(\gamma^2 \geq \bar{p}\)

\[ (42) \]

and that

\[ \arg \max_{N_\gamma \in Q_{\bar{p}}} \min_{\rho_x} f(N_\gamma; \rho_x) = N_{\bar{F}}. \]

\[ (43) \]

The corresponding tradeoff curves are plotted in Fig. 4 for various \(\beta\). The uppermost curves are obtained when \(\beta\) approaches 1, i.e. when \(M_\beta\) is near to the identity (clearly, in this case there is almost no need of inversion). On the other hand, as \(\beta\) goes to zero \(M_\beta\) approaches an orthogonal projector which, in our worst-case criterion, cannot be inverted with nonvanishing minimum fidelity.

4.2. Quantum case

We consider a set of two non-orthogonal states \(D = \{|\psi_\pm\rangle\}\), and we let \(Q\) to be the set of all quantum operations. The states after the first transformation are

\[ |\psi'_\pm\rangle = \frac{M_\beta |\psi_\pm\rangle}{\|M_\beta |\psi_\pm\rangle\|} . \]

\[ (44) \]

The required inversion is then

\[ |\psi'_\pm\rangle \rightarrow |\psi_\pm\rangle . \]

\[ (45) \]

which we already studied in Section 3.

\[ \]
5. Conclusions

After generalizing the state-transformation probability formula of Ref. [13] to mixed target states, we derived a probability-fidelity tradeoff for a varying quantum operation with fixed input-output states. We have then presented the first tradeoff between the probability and the fidelity in the inversion of a quantum operation in a semiclassical and in a quantum case.

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References