Loss-error compensation in quantum-state measurements

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In the two papers [T. Kiss, U. Herzog, and U. Leonhardt, Phys. Rev. A \textbf{52}, 2433 (1995); U. Herzog, Phys. Rev. A \textbf{53}, 1245 (1996)] with titles similar to the one given above, the authors assert that in some cases it is possible to compensate a quantum efficiency $\eta \leq 1/2$ in quantum-state measurements, violating the lower bound 1/2 proved in a preceding paper [G. M. D’Ariano, U. Leonhardt, and H. Paul, Phys. Rev. A \textbf{52}, R1801 (1995)]. Here we reestablish the bound for homodyning any quantum state, and show how the proposed loss-compensation method would fail in a real measurement outside the $\eta > 1/2$ regime.

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In Ref. [1] the homodyne tomography method—so far the only experimental scheme to measure the density matrix of the quantum state of radiation—was shown to be robust to the smearing effect of nonunit quantum efficiency $\eta$ at detectors. It was proved that the measurement is possible only above a lower bound for $\eta$ that depends on the chosen matrix representation of the state $\hat{\rho}$. In particular, for the Fock basis, such a bound is $\eta = 1/2$, and in the same Ref. [1] a reconstruction algorithm that depends parametrically on $\eta$ was provided. As noticed in Ref. [2], the existence of a lower bound for quantum efficiency in state measurements is a relevant issue for the foundations of quantum mechanics, as it prevents the measurement of the wave function of an individual quantum system through a series of repeated weak measurements. Therefore, if a scheme for loss-error compensation is devised, then, in principle, such a scheme can open the door to prove an inconsistency within the logical framework of quantum mechanics.

In Ref. [3] it was proposed to perform the state measurement by using the same algorithm of Ref. [1] for $\eta = 1$, and treating the effect of a low quantum efficiency as a lossy evolution of radiation, thus separating the “bare” detection with $\eta = 1$ from the loss-compensation procedure. More explicitly, the idea is to regard any state measurement with $\eta < 1$ on the “signal” state $\hat{\rho}_{\text{sig}}$ as the corresponding hypothetical bare measurement with $\eta = 1$ on a “dressed” damped state $\hat{\rho}_{\text{meas}}$. Also, stated in different words, the effect of nonunit quantum efficiency is referred to the quantum state itself rather than to the detector, regarding nonunit quantum efficiency in a Schrödinger-like picture, with the state evolving from $\hat{\rho}_{\text{sig}}$ to $\hat{\rho}_{\text{meas}}$, and the quantum efficiency playing the role of a time parameter $t = -\ln \eta$. The core of the method is the inversion of the generalized Bernoulli loss transformation, which relates the matrix elements of the signal state $\hat{\rho}_{\text{sig}}$ with those of the dressed state $\hat{\rho}_{\text{meas}}$ in the Fock basis. This is given by

\[
\langle n | \hat{\rho}_\text{sig} | n + d \rangle = \frac{\eta^{-1/2} (2n + d)^{\infty}}{\sqrt{n!(n + d)!}} \sum_{j=0}^{\infty} \frac{(n + j)! (n + d + j)!}{j!} \times (1 - \eta^{-1})^{n + j} \langle n + j | \hat{\rho}_\text{meas} | n + d + j \rangle.
\]

The argument of Ref. [3] states that above the bound $\eta = 1/2$, the convergence of series (1) is guaranteed for any state, but the convergence radius for $\eta$ depends on the matrix elements of $\hat{\rho}_{\text{meas}}$, and, in principle, for some particular states it is possible for the series to converge also for $\eta < 1/2$. Thus, for example, for a thermal state with $n$ photons, the matrix elements of $\hat{\rho}_{\text{meas}}$ decay as $[\bar{n}/(\bar{n} + 1)]^j$ versus the summation index $j$, and one concludes that the series (1) converges for $\eta > (2 + 1/n)^{-1}$, which violates the bound $\eta = 1/2$. In this Comment we show that this argument does not apply to a measurement, because in this case the series (1) must be evaluated with coefficients given by the measured values of the matrix elements of $\hat{\rho}_{\text{meas}}$ in place of their expectation values $\langle n | \hat{\rho}_{\text{meas}} | n \rangle$. In order to have a successful experiment based on $N$ repeated measurements, the series with measured coefficients must converge for sufficiently large truncation index $j$ to a result within an error that vanishes for large $N$.

For statistically uncorrelated coefficients $c_j$, the variance of the (unconditionally) convergent series $\sum_j^n (\Delta c_j)^2$ is given by the series of variances $\sum_j^n (\Delta c_j^2)$, where $z = 1 - \eta^{-1}$. Then, an \textit{a priori} estimation of the measurement error is given by $\epsilon = \sqrt{\sum_j^n (\Delta c_j^2)} / \sqrt{N}$, with $\epsilon_j = \sqrt{\Delta c_j^2} / \sqrt{N}$. Now, it is clear that statistical errors depend on the particular detection scheme used to measure $c_j = \langle n + j | \hat{\rho}_{\text{meas}} | n + d + j \rangle$. For direct photodetection—which, however, is not a quantum-state measurement—the error $\epsilon_j$ associated to a diagonal element $p_j = \langle n + j | \hat{\rho}_{\text{meas}} | n + j \rangle$ is $\epsilon_j = \sqrt{(1 - p_j)} / \sqrt{N}$, and $\epsilon_j^2$ vanishes linearly with $p_j$ itself for $j \to \infty$. This is the logical basis of the argument of Refs. [3,4] (which we are comment-
only to the loss-error compensation procedure. For this reason, in the following we will always refer to the experiment of homodyne tomography with a state with homodyne detected matrix elements \(\langle j \mid \hat{\rho}_{\text{meas}} \mid j + 2 \rangle\) versus the truncation index \(j_M\). The squares correspond to \(\eta = 0.6\), the circles to \(\eta = 0.55\), the triangles to \(\eta = 0.53\). All points are obtained with Monte Carlo experiments using 24 000 homodyne data.

In Fig. 1 we report the matrix element \(\langle 2 \mid \hat{\rho}_{\text{sig}} \mid 2 \rangle\) for a thermal state with two photons evaluated as the truncated series (1) with homodyne detected matrix elements \(\langle j + 2 \mid \hat{\rho}_{\text{meas}} \mid j + 2 \rangle\) versus the truncation index \(j_M\). The squares correspond to \(\eta = 0.6\), the circles to \(\eta = 0.55\), the triangles to \(\eta = 0.53\). All points are obtained with Monte Carlo experiments using 24 000 homodyne data.

In conclusion, we have shown that the loss-compensation procedure and the same state as in Fig. 1, evaluated as the truncated series (1), as a function of \(\eta\) for different values of the truncation index \(j_M\). The triangles correspond to \(j_M = 10\), the crosses to \(j_M = 20\), and the stars to \(j_M = 100\). All points are obtained with Monte Carlo experiments using 8000 homodyne data.

Below, the result is out of the scale of the figure and oscillates unboundedly versus \(j_M\). In Fig. 2 the statistical error for the same matrix element and the same state as in Fig. 1, evaluated as the truncated series (1), as a function of \(\eta\) for different values of the truncation index \(j_M\): it is apparent that the error converges for all values \(\eta > 1/2\). When the bound \(\eta = 1/2\) is approached, the error starts growing as a function of the truncation index \(j_M\), and diverges for all values \(\eta \leq 1/2\) (the finite slope versus \(\eta\) is only due to the finiteness of \(j_M\)). As we can see from the figure, \(\eta = 1/2\) is manifestly a “transition value” from convergent to divergent behavior of the statistical error.

Actually, in a real experiment homodyne data are contaminated by additional sources of noise other than quantum efficiency. However, for tomography some kinds of noise can still be treated as a negative contribution to the overall quantum efficiency [8].

Before concluding, we want to stress that to our knowledge there is no method—alternative to homodyne tomography—that has statistical errors which make the series (1) convergent for \(\eta < 1/2\). For all methods recently proposed by several authors [9–11], an analysis of statistical errors is still lacking. Moreover, in the set of papers [9] the radiation is measured indirectly through measurements on 

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Notice, however, that the convergence radius $z < \sqrt{R}$ is the square root of the convergence radius of the series (1).

As $k_n \to \infty$ for large values of $n$, $p(x, \phi)$ can be considered constant over a cycle $\Delta x = \pi/k_n$ and the integral over $x$ gives $\epsilon_n = \sqrt{2N}$. A similar argument holds also for off-diagonal elements. This saturation value is overestimated by a factor $\sqrt{2}$ in Ref. [6].