Quantum phase amplification

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A scheme for amplifying phase shifts is presented, based on ideal photon number deamplification. For high-sensitivity measurements phase amplification sizeably reduces the bit-error rate and enhances the mutual information retrieved from the measurement. Phase-coherent states preserve their coherence under amplification, and achieve the best amplifier performance.

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I. INTRODUCTION

In quantum communications the performances of an amplifier depend on the scheme of the channel in which the device is inserted. Noticeably, the added noise is not just intrinsic of the device, but depends on the particular observable that is detected at the receiver. Thus, different kinds of preamplifier work ideally for different kinds of detection [1, 2]: the phase-insensitive amplifier (PIA) is ideal for heterodyning, the phase-sensitive amplifier (PSA) for homodyning, whereas for direct detection different types of devices—heterodyne detection of coherent states. After some remarks, in Sec. VI, on the feasibility of ideal number deamplifiers and phase-coherent state generators, Sec. VII closes the paper.

II. PHASE AMPLIFIERS

The words ‘‘phase amplification’’ can be given a precise meaning in the context of the quantum estimation theory [6]. In our case the problem is the estimation of the phase shift \( \varphi \) of a pure state \( |\psi\rangle \) that undergoes the unitary transformation

\[
|\psi\rangle \rightarrow \exp(ia^\dagger a\varphi)|\psi\rangle, \tag{1}
\]

with \( a^\dagger a \) denoting the number operator of the mode \( a \) of the electromagnetic field. The state \( |\psi\rangle \) itself is assumed to have a well defined phase, say \( \varphi' \), namely, its coefficients in the number representation are of the form

\[
|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle, \quad \psi_n = |\psi_n| \exp(in\varphi'). \tag{2}
\]

Without loss of generality, in the following, we will consider \( \varphi' = 0 \), i.e., all \( \psi_n \) are real non-negative, such that the ideal phase probability distribution \( p(\phi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \psi_n e^{in\phi} \) has both maximum and mean value at \( \phi = 0 \) (averages are evaluated within the \( 2\pi \) periodicity, and are reduced to the \( [-\pi, \pi] \) window). A phase amplifier multiplies the shift \( \varphi \) by a fixed gain \( g \), independently on the state \( |\psi\rangle \). In general, this can be achieved at the expense of introducing some additional noise and of partially destroying the coherence of the input state. In order to avoid biasing, the zero-phase reference of the unshifted state \( |\psi\rangle \) should be retained. This corresponds to a final state that keeps non-negative matrix elements \( \rho_{nm} \geq 0 \ \forall n, m \), because this guarantees that the ideal phase probability distribution \( p(\phi) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \rho_{nm} e^{in\phi} \) still has its maximum at the mean value \( \phi = 0 \), independently on the output state. Therefore, the most general transformation \( A_g \) that describes phase amplification must be of the form

\[
A_g(e^{ia^\dagger a\varphi}\psi) = e^{ia^\dagger a\varphi} \rho_g e^{-ia^\dagger a\varphi}, \tag{3}
\]

with the unbiasedness condition \( \psi_n \geq 0 \ \forall n \Rightarrow (n|\rho_g^{(g)}|m) \geq 0 \ \forall n, m \ [\psi_n \text{ defined in Eq. (2)}] \).
The quantum description of an apparatus for detecting a phase shift is given by a Born’s rule in the following general form:

\[ p(\phi | \phi) \, d\phi = \text{Tr}[e^{i\phi a^\dagger a} | \phi \rangle \langle \phi | e^{-i\phi a^\dagger a} \hat{d}(\phi)], \]

where \( p(\phi | \phi) \) is the probability density of detecting a phase shift \( \phi \) “conditioned” by the actual value \( \phi \), whereas \( \hat{d}(\phi) \) denotes the POM (probability operator-valued measure) [6] that pertains to the apparatus. When there is no \( a \) priori preferred phase, the conditional probability density should have the form \( p(\phi | \phi) = p(\phi - \phi) \), namely, it should depend only on the difference between the detected and the actual value of the phase shift. In this case the POM has the covariant form

\[ d\hat{d}(\phi) = e^{i\phi a^\dagger a} \xi e^{-i\phi a^\dagger a} \frac{d\phi}{2\pi}, \]

with \( \xi \) being a fixed positive operator. For unbiased measurements (i.e., \( \phi \) equals the averaged \( \phi \)) all matrix elements of the operator \( \xi \) are non-negative in the number representation [7,8]. For heterodyne or double homodyne [9] phase detection, the phase shift is retrieved from the phase of the output complex photocurrent. In this case the operator \( \xi \) is given by [10]

\[ \xi = \sum_{n,m=0}^{\infty} |n\rangle \langle m| \frac{\Gamma[(\frac{1}{2}(n+m)+1)]}{\sqrt{n!m!}}. \]

On the other hand, quantum estimation theory allows optimization of the POM in Eq. (4) at a purely abstract level, providing the theoretical description of an ideal phase measurement. For covariant measurements the ideal POM has operator \( \hat{\xi} \) given by [11,12]

\[ \hat{\xi} = \sum_{n,m=0}^{\infty} |n\rangle \langle m|. \]

There is, however, no known apparatus that would approach such ideal POM.

In terms of the detector’s POM a phase amplifier must achieve the dual transformation of map (3), namely,

\[ A_\xi^\dagger[d\hat{d}(\phi)] = A_\xi^\dagger\{e^{i\phi a^\dagger a} \xi e^{-i\phi a^\dagger a} \} \frac{d\phi}{2\pi} = e^{i\phi a^\dagger a} \xi e^{-i\phi a^\dagger a} \frac{d\phi}{2\pi}, \]

where \( \langle n|\hat{\xi}|m\rangle = |n\rangle \langle m| \hat{\xi} |n\rangle \langle m| \to 0 \forall n,m \), and duality is defined through the identity of traces \( \text{Tr}(A^\dagger B) = \text{Tr}[\hat{A}\hat{A}(\hat{B})] \) with \( \hat{A} \) bounded and \( \hat{B} \) trace-class. In fact, the POM transformation (8) is the Heisenberg-picture form of the Schrödinger-picture map (3), and gives the POM of the total detector, including the preamplification. Equation (8) is the only transformation that assures that the conditional probability (4) after amplification is just a function of \( g \phi - \phi \) for every \( \phi \).

III. IDEAL NUMBER DEAMPLIFICATION AND PHASE AMPLIFICATION

We are now in position to understand how an ideal PNA can also achieve ideal phase amplification. In Ref. [13] the Hamiltonian of the ideal PNA is derived, showing that such a device is “canonical” for the number-phase couple—a Fourier-transform conjugated pair [14]. By “canonical” we denote a device analogous to the PSA, where a quadrature is amplified while the conjugated one is simultaneously deamplified. Here the PNA, when used in the inverse way as an ideal number deamplifier, works also as a phase amplifier. Since the photon number is an integer, ideal number amplification and deamplification are attained only for integer values of the gain \( g \) [15]. Ideal number deamplification and simultaneous phase amplification are described by the unitary operator [13]

\[ \hat{U}_g = \sum_{n} \sum_{m=0}^{\infty} |n\rangle \langle gn+v| \otimes |gm+v\rangle \langle m| \]

that acts on the enlarged Hilbert space \( \mathcal{H} \otimes \mathcal{H}_f \) including the signal Hilbert space \( \mathcal{H} \) and the space \( \mathcal{H}_f \) of an additional idler mode. An auxiliary idler mode is needed in order to preserve unitarity [16]: as we will see in the following, the idler mode is responsible for “mixing” the state, as in Eq. (3). The amplifying maps \( A_g \) and \( A_g^\dagger \) pertain to the signal Hilbert space \( \mathcal{H} \) only, and are obtained by partially tracing over the idler mode. One has

\[ A_g((|\psi\rangle \langle \psi|) = \text{Tr}_{\hat{f}}[\hat{U}_g |\psi\rangle \langle \psi| \otimes \hat{\rho}_f \hat{U}_g^\dagger], \]

\[ A_g^\dagger[d\hat{d}(\phi)] = \text{Tr}_{\hat{f}}[\hat{U}_g d\hat{d}(\phi) \otimes \hat{U}_g^\dagger \hat{1} \otimes \hat{\rho}_f], \]

with \( \hat{\rho}_f \) denoting the density matrix of the idler mode. From Eqs. (9) and (11) one can see that the ideal number deamplifier achieves the phase amplification given in Eq. (8) for any \( \hat{\rho}_f \), with

\[ \hat{\xi}_g = g^{-1} \sum_{k=0}^{\infty} \xi_g^{(k)}, \]

\[ \xi_g^{(k)} = \sum_{n,m=0}^{\infty} 2\pi e^{i\pi g^{-1}(n-m)}\langle n|[(n/g)] \xi[|m/g]\rangle \langle m|, \]

where \( [x] \) denotes the integer part of \( x \). The sum over \( \lambda \) accounts for the \( 2\pi \) periodicity. In fact, one has

\[ A_g[d\hat{d}(\phi)] = \frac{d\phi}{2\pi g} \sum_{x=0}^{g^{-1}} e^{i\phi a^\dagger a^{-1}(\phi + 2\pi x \lambda)} \xi^{(0)}_g \times e^{-i\phi a^\dagger a^{-1}(\phi + 2\pi \lambda)}, \]

with \( \xi^{(0)}_g \) given by Eq. (13). Notice that for the ideal POM (7) one has \( \xi^{(0)}_g = \hat{\xi}_g \), and Eq. (14) is just a \( 2\pi \)-periodic rescaling of the POM: in this sense the present phase amplification can be considered ideal. It is also easy to check that the amplifier achieves the Schrödinger-picture transformation (3) independently on the idler state \( \hat{\rho}_f \). One has
\[ p^{(g)}_{\phi} = \sum_{j=0}^{g-1} |\psi_{\phi}^{(j)}\rangle\langle\psi_{\phi}^{(j)}|, \quad |\psi_{\phi}^{(j)}\rangle = \sum_{n=0}^{\infty} \psi_{n+j} |n\rangle, \]  

where, for simplicity of notation, we retain unnormalized vectors $|\psi_{\phi}^{(j)}\rangle$.

**IV. AMPLIFICATION OF PHASE-COHHERENT STATES**

From Eq. (15) it follows that $p^{(g)}_{\phi}$ is pure only when $\psi_{n} \propto r^{n}$ for some constant $r$; hence, the only states which are not mixed by phase amplification—so providing optimum performance—are the so-called phase-coherent states [14]. These are defined as follows:

\[ |\xi\rangle = (1 - |\xi|^2)^{1/2} \sum_{n=0}^{\infty} \xi^n |n\rangle, \quad \xi = e^{i\varphi} |\xi|, |\xi| < 1, \]  

where the complex number $\xi$ also carries the phase-shift information $\varphi$. Then, from Eqs. (10), (15), and (16) one has

\[ \mathcal{A}_\varphi(|\xi\rangle\langle\xi|) = |\xi^\varphi\rangle\langle\xi|, \]  

The phase-coherent state $|\xi\rangle$ has an average number of photons $\langle n\rangle = |\xi|^2/(1 - |\xi|^2)$. Notice that, apart from normalization, in the limit $|\xi| \to 1$ the state (16) approaches the Susskind-Glogower state

\[ |e^{i\varphi}\rangle = \sum_{n=0}^{\infty} \exp(in\varphi) |n\rangle, \]  

in terms of which the ideal POM can be rewritten as $d\mu(\phi) = |e^{i\phi}\rangle\langle e^{i\delta}| d\phi/2\pi$; in this sense one can say that the phase-coherent states match the ideal POM for large number of photons. For ideal phase detection, the output phase probability after amplification is simply given by

\[ p_{\text{out}}(\phi|\phi) = \frac{1}{2\pi} \left( |\xi^\varphi|^2 \right)^2 \]  

\[ = \frac{1}{2\pi} \frac{1 - |\xi|^2 e^{2g}}{1 + |\xi|^2 e^{-2g} - 2|\xi|^2 e^{g} \cos(\phi - g \varphi)}. \]  

In the limit $|\xi| \to 1$ one has $p_{\text{out}}(\phi) \to 2\pis_2\pi(\phi - g \varphi)$, $\delta_{2\pi}$ denoting the periodicized $\delta$. All quantities of interest can be evaluated analytically for $|\xi| = 1 - \varepsilon$ with $g \varepsilon \ll 1$ and $g \varphi \in [-\pi, \pi]$. The average phase is amplified as

\[ \langle \phi \rangle_\text{out} = g \varphi + O(g \varepsilon), \]  

whereas the root-mean-square (rms) fluctuations

\[ \langle \Delta \phi^2 \rangle_\text{out} = 2g \varepsilon + O(g^2 \varepsilon^2) \]  

are amplified by only a factor $g$. Thus if we define the noise figure

\[ R = \frac{\langle S^2/N \rangle_\text{in}}{\langle S^2/N \rangle_\text{out}}, \]  

where $S$ and $N$ denote, respectively, the signal $\langle \phi \rangle$ and the noise $\langle \Delta \phi^2 \rangle$, we have

**V. PHASE MEASUREMENTS BASED ON BINARY HYPOTHESIS TESTING**

Typically, the situation in which one takes advantage of amplification occurs when the signal is very low, below the detection threshold, and the amplifier is used to enhance the signal above the threshold. However, as amplification also increases noise, the net benefit must be evaluated carefully, by comparing the values of BER and mutual information [19] obtained with and without amplification. A paradigmatic situation is sketched in Fig. 2, where we consider a binary channel that pertains to the phase detection of a small signal, i.e., a small phase shift. The measurement consists of testing the hypothesis that a phase-shifting event has occurred, assigning the "true" value to every outcome above the threshold $\varphi_s$. The phase probability distributions corresponding to zero shift and to $\varphi$ shift are depicted in different gray colors; they correspond to the reference zero-phase state $|\cdot'0\cdot'\rangle = |\psi\rangle$ and to the shifted state $|\cdot'1\cdot'\rangle = \exp(i\alpha |\text{a}\rangle |\varphi\rangle) |\psi\rangle$, respectively. The input signal is
very weak ($\varphi \ll 1$): the threshold $\varphi_s$ is taken above $\varphi$ due to limitations of the detector sensitivity and in order to achieve a low value of the “false alarm probability” $Q_{1|0}$ [6,19]

$$Q_{1|0} = \int_{\varphi_s}^{\pi} d\phi p(\phi|0),$$

namely, the probability of detecting “1” given state $|\cdot\cdot0\cdot\cdot\rangle$. It is clear that amplification will increase $Q_{1|0}$ as a consequence of the spread of the right tail of the “0” distribution; however, it will simultaneously enhance the “detection probability” $Q_{1|1}$ [6,19]

$$Q_{1|1} = \int_{\varphi_s}^{\pi} d\phi p(\phi|\varphi),$$

namely, the probability that “1” is correctly detected given state $|\cdot\cdot1\cdot\cdot\rangle$. An improvement of the binary test measurement is determined by a decrease of the bit-error rate

$$B = 1 + Q_{1|0} - Q_{1|1},$$

or, equivalently, by an enhancement of the mutual information [19]

$$I = \sum_{j,k=0}^{1} p_j Q_{k|j} \ln \frac{Q_{k|j}}{\sum_{i=0}^{1} p_i Q_{k|i}}.$$

These probabilities give the BER and the mutual information plotted in Fig. 3 as a function of the gain for different values of the input number of photons $\langle \hat{n} \rangle_i$. The case of a very weak input signal $\varphi \ll \varphi_s$ has been considered. One can see that the BER exhibits a steep decrease and that, at the same time, the
mutual information shows a rapid increase near the gain \( g_s = \varphi_s / \varphi \). These features are further enhanced when the mean input photon number is increased. For the mutual information we refer to the situation of rare events, i.e., \( p_1 = 1 - p_0 \ll 1 \), which is of interest, for example, in interferometric detection of gravitational waves: the behavior of \( I \), however, does not qualitatively depend on \( p_1 \), apart from the range of variation. On the other hand, if the input signal is above the detection threshold, i.e., \( \varphi > \varphi_s \), one could see that the mutual information would monotonically decrease versus \( g \), whereas there would be essentially no reduction of the BER. This is due to the fact that in this case amplification decreases the detection probability \( Q_{1|1} \) given in Eq. (29).

**B. Coherent states**

Phase-coherent states are the only coherence preserving states under phase amplification. For generic input states phase amplification changes the kind of state and partially destroys coherence: for example, phase amplification does not preserve coherent or squeezed states. However, this does not mean that for such states the amplifier cannot improve the phase-shift measurement (on this subject, a preliminary indication is found in Fig. 1). Especially for nonideal phase detection, one can gain much benefit from phase amplification, also because the amplifier partially recovers the effective loss due to nonideal measurement. As an example, in Fig. 4 we have considered the realistic case of heterodyne phase detection of coherent states: here, the BER and the mutual information are plotted in the same fashion of Fig. 3 and for the same values of parameters \( \varphi, \varphi_s \), and \( \langle \hat{n} \rangle_{in} \). One can see that the amplifier works effectively, almost as well as for phase-coherent states. The only negative features are that the variations of \( B \) and \( I \) are less steep, and the amplifier efficiency is much reduced for low numbers of input photons. These phenomena are distinctive of a partial loss of coherence of the amplified state.

We emphasize that phase amplification is advantageous only for measurements of small phase shifts \( \varphi \), and not too large gains \( g \), such that \( g \varphi \ll 1 \). In fact, the transformation (14) folds the probability distribution at the boundaries of the \( 2\pi \) window in order to maintain the distribution as \( 2\pi \) periodic after the stretching along the direction of abscissa. In this way, in the limit of large gains any probability distribution would converge to the uniform probability on the \( 2\pi \) window.

In conclusion of this section some comments are in order regarding the apparent violation of the data processing theorem [19] regarding the improvement of mutual information. Indeed the theorem states the impossibility of improving the mutual information by performing any kind of data processing. More precisely, for a channel described by a map \( X \rightarrow Y \) between input-output random variables \( X \) and \( Y \), the mutual information \( I(X|Y) \) between \( X \) and \( Y \) cannot be improved neither by any kind of “encoding” \( U \rightarrow X \), nor by any “decoding” \( Y \rightarrow V \), where \( U \) and \( V \) are additional random variables. In other words: \( I(U|V) \ll I(X|Y) \), i.e., the end-to-end mutual information of the long Markov chain \( U \rightarrow X \rightarrow Y \rightarrow V \) is never greater than that of the short chain \( I(X|Y) \). The data processing theorem does not pertain to the present case of insertion of a quantum amplifier in a channel for the following two reasons. On the one hand, the amplifier is not used neither as an encoder, nor as a decoder—i.e., at one of the two ends of the chain—but is inserted in the chain as a *preamplifier* before a source of additive noise. If the amplifier admits a classical description in terms of an input-output probability map, then the insertion of the amplifier would correspond to changing the Markov chain \( X \rightarrow Y \) to \( X \rightarrow V \rightarrow Y \)—namely, to changing the map \( X \rightarrow Y \) instead of adding another data processing at one end of the chain: hence, the conditions for the data processing theorem do not apply. On the other hand, the amplifier is not a “classical” data processor, i.e., it is not equivalent to a measurement followed by data processing: coherence is only partially destroyed throughout the amplification process, and hence the amplifier is described by a map between probability amplitudes, rather than by a map between input-output probabilities. Probabilities are determined only at the very end of the chain, and depend on the observable that is measured at the output. In the quantum description, in addition to the input-output random variables \( X \) and \( Y \) we need to specify the detection POM \( d\tilde{\mu}(Y) \) at the end of the channel and the quantum state \( \hat{\rho}_X \) encoding the input variable \( X \), such that the probability map \( X \rightarrow Y \) is given by the output conditional

![Fig. 4. Bit-error rate (a) and mutual information (b) vs gain g for coherent states and heterodyne detection (same values of parameters as in Fig. 3).](image-url)
probability density \( p(Y|X)dY = \text{Tr}[\hat{\rho}_X d\hat{\mu}(Y)] \). Hence, the mutual information of the quantum channel can be denoted as
\[
I[X, \hat{\rho}_X; Y, d\hat{\mu}(Y)].
\]
The insertion of a device in the quantum channel is described by a trace-preserving completely positive (CP) map \( \hat{\rho}_X \rightarrow \mathcal{A}(\hat{\rho}_X) \) or by its dual \( d\hat{\mu}(Y) \rightarrow A^\dagger(d\hat{\mu}(Y)) \). In general, if the system is suboptimal (i.e., the information is not optimized over the detection POM), there is no fixed inequality between
\[
I[X, \hat{\rho}_X; Y, d\hat{\mu}(Y)] \quad \text{and} \quad I[X, A(\hat{\rho}_X); Y, d\hat{\mu}(Y)]:
\]
the quantum device described by the CP map \( A \) reshapes the channel (i.e., leads to a different conditional probability between \( X \) and \( Y \)) with the possibility of improving the mutual information and approaching conditions for optimality. This situation corresponds to our case, where the system is suboptimal. If one wants to recover a situation closer to the one of the classical data processing theorem, one should optimize the mutual information over the detection POM, and the quantum analog of the data processing theorem can be written as follows:
\[
\max_{d\hat{\mu}(Y)} I[X, A(\hat{\rho}_X); Y, d\hat{\mu}(Y)]
\]
\[
\leq \max_{d\hat{\mu}(Y)} I[X, \hat{\rho}_X; Y, d\hat{\mu}(Y)].
\]
(30)
The measurement-optimized system then cannot be further improved by the insertion of another device.

VI. EXPERIMENTAL REALIZATION OF THE SCHEME

Before concluding, some comments regarding the generation of phase-coherent states and the practical feasibility of photon number deamplifiers are in order.

Phase-coherent states can be ideally achieved using a PIA and a PND in series, as shown in Ref. [20]. In fact, the unitary evolution operator of the PIA is
\[
\hat{U}_{\text{PIA}} = \exp[\xi a^\dagger b^\dagger - \xi ab],
\]
where \( a \) and \( b \) describe signal and idler modes. When both modes are in the vacuum state at the input, the state at the output is
\[
\hat{U}_{\text{PIA}}(|0\rangle \otimes |0\rangle) = (1 - |\xi|^2)^{1/2} \sum_{n=0}^{\infty} \xi^n |n\rangle \otimes |n\rangle,
\]
where \( \xi = \xi |\text{tanh} |\xi | \). Then the twin beams are ideally converted into the one-mode phase-coherent state (16) using the PND in the inverse way. Concretely, the PND evolution can be well approximated by a sum-frequency up converer, described by the interaction Hamiltonian
\[
\hat{H} = k(abc^\dagger + a^\dagger b^\dagger c).
\]
For \( c \) initially in the vacuum state the performance of the sum-frequency converter very nearly approaches an ideal inverse PND [21]. The best approximation corresponds to maximum conversion for the mean photon number from

![FIG. 5. Phase probability distribution of a phase-coherent state compared with the probability of a state achieved using a PIA and a sum frequency up converter in series (the sharper probability refers to the ideal state). The resulting average photon number is \( \langle \hat{n} \rangle = 9.26 \).](image)

VII. CONCLUSIONS

In conclusion, we have proposed a scheme for amplifying small phase shifts which reduces the BER and increases the information retrieved from the measurement. The best performance is achieved by phase-coherent states, but good results are also obtained in the practical situation of coherent states with heterodyne phase detection. We have shown how the PNA and PND—both originally proposed for matching direct detection—can be profitably used also for phase detection. When used as an ideal number deamplifier, the PNA becomes a phase amplifier that achieves ideal amplification independently on the state of the idler mode. The feasibility of both phase-coherent state generation and ideal number deamplification has been analyzed, based on phase-insensitive amplification, sum-frequency up conversion, and g-harmonic generation.

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In fact, from Eq. (5) it follows that the probability distribution of any covariant phase detector is equivalent to the ideal one \( p(\phi) = \frac{1}{2\pi} \sum_{n,m=0}^{\infty} \rho_{nm} e^{i(n-m)\phi} \), but for a different state \( \hat{\rho}' = C(\hat{\rho}) \), where \( \rho_{nm}' = \rho_{nm} \xi_{nm} \) (\( C \) is a trace-preserving completely positive map). Hence, the condition \( \xi_{nm} \geq 0 \) \( \forall n,m \), assures that \( \rho_{nm}' \geq 0 \) \( \forall n,m \) independently on the state.

Notice the difference between: (i) non-negative operator and (ii) operator having only non-negative matrix elements in the number representation. Case (i) means that all diagonal matrix elements are non-negative in any representation. Hence, in general it is possible to have operators satisfying (ii) but not (i) and, conversely, operators satisfying (i) but not (ii).

We use squeezed states with a squeezing fraction sufficiently low such that tails of the phase probability distribution at \( \varphi + \pi \) are negligible.

The integer nature of the number operator leads to number fluctuations related to the fractional part of the division of the photon number by the integer gain \( g \). However, this additional noise becomes negligible for input photon numbers \( \langle \hat{n} \rangle > g \) (see Ref. [13]).

The need of an auxiliary idler mode is a common feature of all amplifiers, with the exception of the PSA.

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The POM is optimized according to some goodness criterion, usually by minimizing the average cost for some “cost function.” The ideal POM (7) minimizes the average cost for a large class of cost functions, including those corresponding to max-likelihood criterion and minimum 2\( \pi \)-periodic rms noise: see [11].

References

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