Estimation of squeezed state

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Main purpose is finding optimal estimator for squeezed state:

and evaluating its performance.

Contents

- Properties of squeezed state
- Derivation of the optimal measurement with n copies
- Second order asymptotics
- Cloning

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Boson Fock space

Boson Fock space

a **: anihilation operator**

$a^\dagger$ **: creation operator**

$N \triangleq a^\dagger a$ **: number operator**

$|n\rangle_N$ **: number state (**$N|n\rangle_N = n|n\rangle_N$ **),**

$|\alpha\rangle_a$ **:Boson coherent state (**$a|\alpha\rangle_a = \alpha|\alpha\rangle_a$ **),**

Squeezed state

Caves's notation

$$\left|\alpha;\xi\right\rangle_c \triangleq \exp\left(-\frac{\xi}{2}\left(a^\dagger\right)^2+\frac{\bar{\xi}}{2}a^2\right)\left|\alpha\right\rangle_a$$

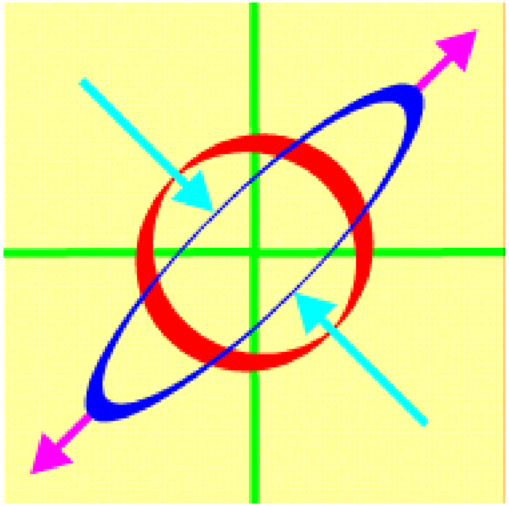
Yuen's notation

$$\left(\mu a+\nu a^\dagger\right)\left|\alpha;\mu,\nu\right\rangle_y=\alpha\left|\alpha;\mu,\nu\right\rangle_y,$$

where $\left|\mu\right|^2-\left|\nu\right|^2=1.$

$$\left|\alpha;\xi\right\rangle_c=\left|\alpha;\cosh\left|\xi\right|,e^{i\arg\xi}\sinh\left|\xi\right|\right\rangle_y.$$

Another Parameterization of Squeezed state



In the case of α=0, we have

$$\left|0; e^{i\theta} t\right\rangle_c$$

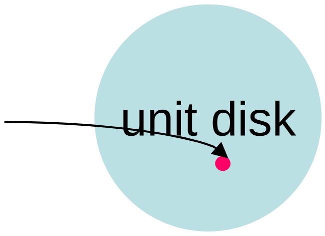
$$= \left|0; \cosh t, e^{i\theta} \sinh t\right\rangle_y .$$

Parameterize

As another expresssion, we focus

on $\left|e^{i\theta} r\right\rangle_s \triangleq \left|0; e^{i\theta} \tanh^{-1} r\right\rangle_c ,$

$e^{i\theta} r$



$e^{i\theta} r \in D$ **: unit disk.**

θ Angle of squeezing

r Scale of squeezing

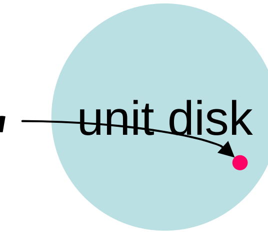
This is useful for group covariant method.

Merit of the expression $|\beta\rangle_s$

Since $\left(\mu a + \nu a^\dagger\right)|0;\mu,\nu\rangle_y = 0,$

$$-\left(a^\dagger\right)^{-1} a|0;\mu,\nu\rangle_y = \frac{\nu}{\mu}|0;\mu,\nu\rangle_y .$$

$e^{i\theta}r$



$\left(a^\dagger\right)^{-1}$ **is defined as the inverse of**

$a^\dagger : L^2_{\text{even}}(\mathbb{R}) \to L^2_{\text{odd}}(\mathbb{R}).$

Note that $|0;\mu,\nu\rangle_y \in L^2_{\text{even}}(\mathbb{R}).$

That is, $-\left(a^\dagger\right)^{-1} a|0;e^{i\theta}t\rangle_c = e^{i\theta}\tanh t\,|0;e^{i\theta}t\rangle_c .$

In other word, $-\left(a^\dagger\right)^{-1} a|\beta\rangle_s = \beta|\beta\rangle_s ,$

$|e^{i\theta}r\rangle_s = |0;e^{i\theta}\tanh^{-1} r\rangle_c ,\ e^{i\theta}r \in D$ **: unit disk.**

n copies case of squeezed state

In coheret state case,

$$\frac{a_1+\cdots a_n}{n}\left|\alpha\right\rangle_a^{\otimes n}=\alpha\left|\alpha\right\rangle_a^{\otimes n}.$$

In squeezed state case,

letting $a^{(n)} \triangleq -\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n}a_i^{\dagger}a_i,$

we have $a^{(n)}\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\frac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n},$

$i.e.,$ $a^{(n)}\left|\beta\right\rangle_s^{\otimes n}=\beta\left|\beta\right\rangle_s^{\otimes n}.$

Proof of $a^{(n)}\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\dfrac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n}$

$$\left(\mu a_i+\nu a_i^{\dagger}\right)\left|0;\mu,\nu\right\rangle_y^{\otimes n}=0,$$

$$i.e.,-\mu a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\nu a_i^{\dagger}\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$$

Thus, $-\mu a_i^{\dagger}a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\nu\left(a_i^{\dagger}\right)^2\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$

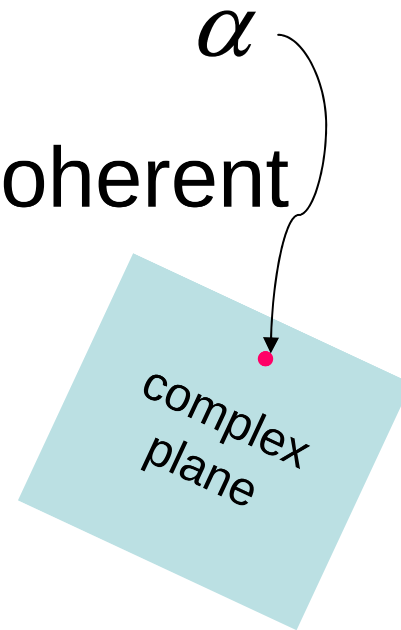
Hence, $-\mu\sum_{i=1}^{n}a_i^{\dagger}a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\nu\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$

Therefore,

$$-\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n}a_i^{\dagger}a_i\left|0;\mu,\nu\right\rangle_y^{\otimes n}=\frac{\nu}{\mu}\left|0;\mu,\nu\right\rangle_y^{\otimes n}.$$

Coherent state

- The annihilation operator $\boldsymbol{a}$ and coherent state $|\alpha\rangle_{\boldsymbol{a}}$ satisfy $\boldsymbol{a}|\alpha\rangle_{\boldsymbol{a}} = \alpha|\alpha\rangle_{\boldsymbol{a}}$.



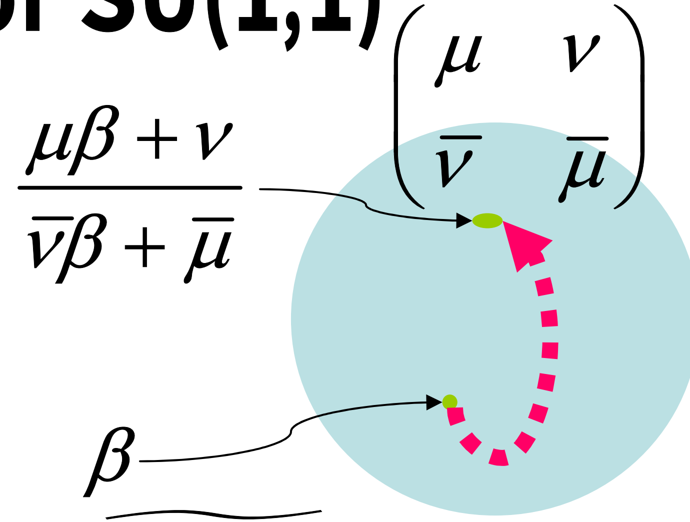
- The heterodyne measurement $\boldsymbol{M}(\boldsymbol{d}\hat{\alpha}) \triangleq \frac{\mathbf{1}}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}\ \boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha}$ satisfies

$$\boldsymbol{a} = \int_{\mathbb{C}} \frac{\hat{\alpha}}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}\ \boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha},\ \boldsymbol{a}\boldsymbol{a}^\dagger = \int_{\mathbb{C}} \frac{|\hat{\alpha}|^2}{\mathbf{2}\pi}|\hat{\alpha}\rangle_{\boldsymbol{a}\ \boldsymbol{a}}\langle\hat{\alpha}|\boldsymbol{d}^2\hat{\alpha}.$$

- The heterodyne measurement is the optimal estimator of the family $\left\{|\alpha\rangle_{\boldsymbol{a}} \middle| \alpha \in \mathbb{C}\right\}$.

Similar properties are expected for squeezed state $|\beta\rangle_{\boldsymbol{s}}$ and operator $\boldsymbol{a}^{(1)} = -\left(\boldsymbol{a}^\dagger\right)^{-1}\boldsymbol{a}$ or $\boldsymbol{a}^{(n)}$.

The action of $\widetilde{SU(1,1)}$: double-covering group of SU(1,1)



The representation V **of** $\widetilde{\text{SU}(1,1)}$ **is given as follows.**

$$V(g)aV(g)^{\dagger} = \mu a + \nu a^{\dagger},$$

$\forall g \in \widetilde{\text{SU}(1,1)}$, **where**

$\tilde{\pi}(g) = \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix}$ **is the projection from** $\widetilde{\text{SU}(1,1)}$ **to SU(1,1).**

This representation acts on the squeezed state as

$$V(g)\left|\beta\right\rangle_{s\ s}\left\langle\beta\right|V(g)^{\dagger} = \left|\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}\right\rangle_{s\ s}\left\langle\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}\right|.$$

The invariant mearsure on the unit disk D **is** $\dfrac{d^2\beta}{\pi\left(1-|\beta|^2\right)^2}$.

Group covariace by $\widetilde{SU(1,1)}$ in the n - copy case

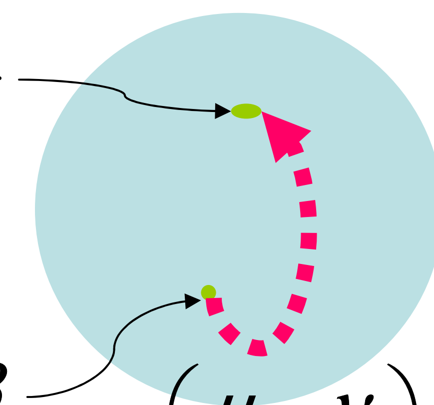
$$\rho_\beta \triangleq |\beta\rangle_{s\ s}\langle\beta|,\ \beta \in D \triangleq \{z \in \mathbb{C} \,\|z| < 1\},$$

We focus on the state family $\left\{\rho_\beta^{\otimes n} \,\middle|\, \beta \in D\right\}$

with the following action.

$$V(g)^{\otimes n} \rho_\beta^{\otimes n} \left(V(g)^{\otimes n}\right)^\dagger = \rho_{\frac{\mu\beta+\nu}{\bar{\nu}\beta+\bar{\mu}}}^{\otimes n},$$

where $\pi(g) = \begin{pmatrix} \mu & \nu \\ \bar{\nu} & \bar{\mu} \end{pmatrix},$



π **is the projection from** $\widetilde{SU(1,1)}$ **to SU(1,1).**

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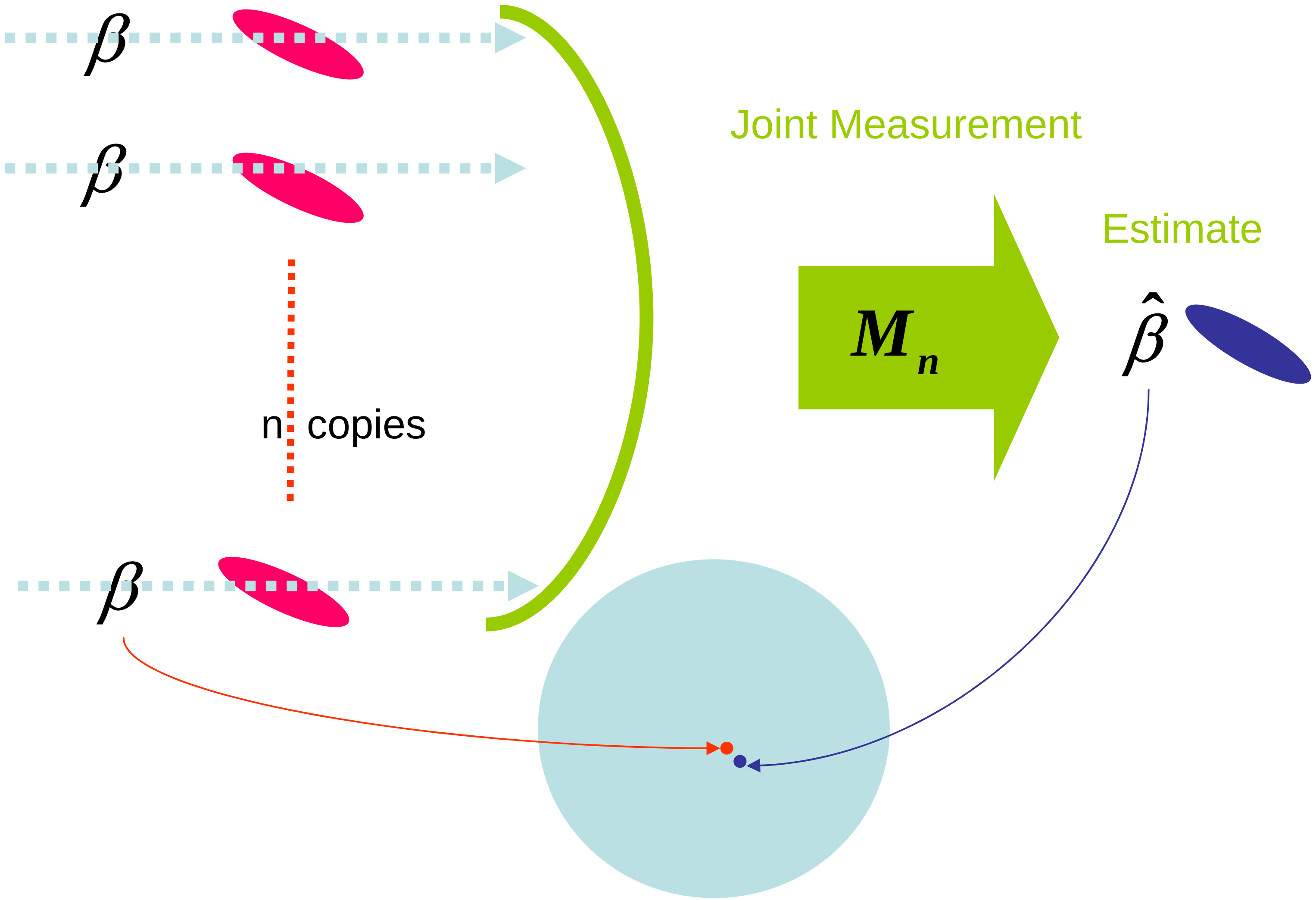
Optimal estimator



Optimal estimator

$n > 2$, The error function $W(\beta, \hat{\beta})$ is a monotone decreasing function of the fidelity:

$$\left| {}_s\langle \beta | \hat{\beta} \rangle_s \right|^2 = \left(1 - \left| \frac{\beta - \hat{\beta}}{\bar{\beta}\hat{\beta} - 1} \right|^2 \right)^{\frac{1}{2}}.$$

The optimal measurement is

$$M_n(d^2\hat{\beta}) \triangleq \left(\frac{n}{2} - 1 \right) \left(|\hat{\beta}\rangle_s \, {}_s\langle \hat{\beta}| \right)^{\otimes n} \frac{d^2\hat{\beta}}{\pi(1 - |\hat{\beta}|^2)^2}.$$

Its optimal distribution is

$$\mathrm{Tr} M_n(d^2\hat{\beta}) \rho_\beta^{\otimes n} = \left(\frac{n}{2} - 1 \right) \left(1 - \left| \frac{\beta - \hat{\beta}}{\bar{\beta}\hat{\beta} - 1} \right|^2 \right)^{\frac{n}{2}} \frac{d^2\hat{\beta}}{\pi(1 - |\hat{\beta}|^2)^2}.$$

Framework of group covariant estimation

Assume that the state family $\left\{ \rho_\theta \middle| \theta \in \Theta \right\}$ **and group representation** V **of the grpup** G **satisfy that** $V(g)\rho_\theta V(g)^\dagger = \rho_{\pi(g)\theta}$, **where** π **is the action of** G **to** Θ.

Suppose that the error function $W(\theta,\hat{\theta})$ **satisfies** $W(\theta,\hat{\theta}) = W(\pi(g)\theta,\pi(g)\hat{\theta})$, *e.g.*, $1-$ **fidelity** *etc*.

$$D_\theta^W(M) \triangleq \int_\Theta W(\theta,\hat{\theta}) \mathrm{Tr} M(d\hat{\theta})\rho_\theta.$$

Minimax method: minimize $D^W(M) \triangleq \sup_{\theta\in\Theta} D_\theta^W(M)$.

Quantum Hunt-Stein's lemma:

$$\min_M D^W(M) = \min_{M:\mathrm{cov}} D_\theta^W(M).$$

M **is covariant if** $M(\pi(g)d\hat{\theta}) = V(g)M(d\hat{\theta})V(g)^\dagger$.

Optimal performance

Fidelity:

$$\int_D \left| {}_s\langle \beta | \hat{\beta} \rangle_s \right|^2 \mathrm{Tr} M_n(d^2\hat{\beta}) \rho_\beta^{\otimes n}$$

$$= 1 - \frac{1}{n-1} \approx 1 - \frac{1}{n} + (1-2)\frac{1}{n^2}.$$

Square of Bures' distance:

$$\int_D \left(1 - \left| {}_s\langle \beta | \hat{\beta} \rangle_s \right| \right) \mathrm{Tr} M_n(d^2\hat{\beta}) \rho_\beta^{\otimes n}$$

$$= \frac{2}{2n-3} \approx \frac{1}{n} - \left(\frac{1}{2} - 2 \right) \frac{1}{n^2}.$$

Relation to the operator $a^{(n)}$

The measurement

$$M_n(d^2\beta) \triangleq \left(\frac{n}{2}-1\right)\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\frac{d^2\beta}{\pi(1-|\beta|^2)^2}$$

and the opeartor $a^{(n)} \triangleq -\left(\sum_{i=1}^{n}\left(a_i^{\dagger}\right)^2\right)^{-1}\sum_{i=1}^{n}a_i^{\dagger}a_i$

satisfiy the property similar to coherent case, $i.e.$**,**

$$a^{(n)} = \int_D \beta M_n(d^2\beta),\ a^{(n)}a^{(n)\dagger} = \int_D |\beta|^2 M_n(d^2\beta).$$

$$\because \quad a^{(n)} = \int_D a^{(n)} M_n(d^2\beta) = \int_D \beta M_n(d^2\beta).$$

$$a^{(n)}a^{(n)\dagger} = \int_D a^{(n)} M_n(d^2\beta) a^{(n)\dagger}$$

$$= \int_D \beta M_n(d^2\beta)\overline{\beta} = \int_D |\beta|^2 M_n(d^2\beta).$$

Case of n=1, n=2

c.f. Optimal POVM

In the case of $n=1,2$,

$$\int_D \left(|\beta\rangle_s\ {}_s\langle\beta|\right)^{\otimes n} \frac{d^2\beta}{\pi(1-|\beta|^2)^2} = \infty \cdot \left(|\beta\rangle_s\ {}_s\langle\beta|\right)^{\otimes n} \frac{\left(\frac{n}{2}-1\right) d^2\beta}{\pi(1-|\beta|^2)^2}$$

Hence, there is no optimal covariant measurement.

However, in the case of $n=2$, there exsists

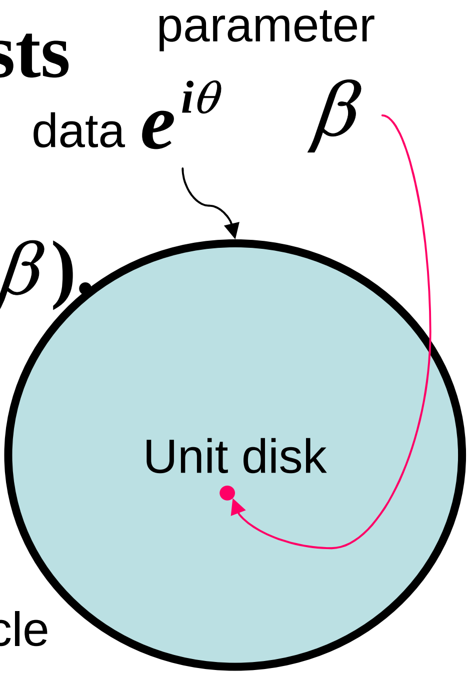
a POVM M_2 such that

$$a^{(2)} = \int_U \beta M_2(d\beta),\ a^{(2)} a^{(2)\dagger} = \int_U |\beta|^2 M_2(d\beta),$$

the POVM M_2 has the measuring data

in the unit circle $U \triangleq \{z \in \mathbb{C} \,\|\, z| = 1\}$.

Note that it is out of parameter space.



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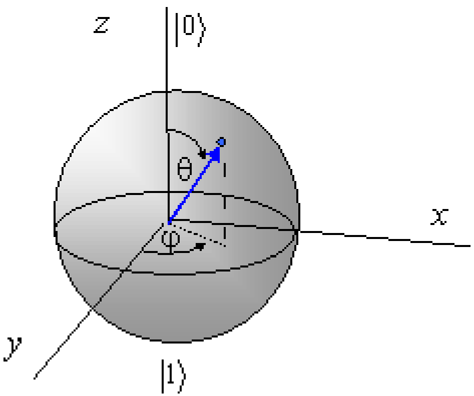
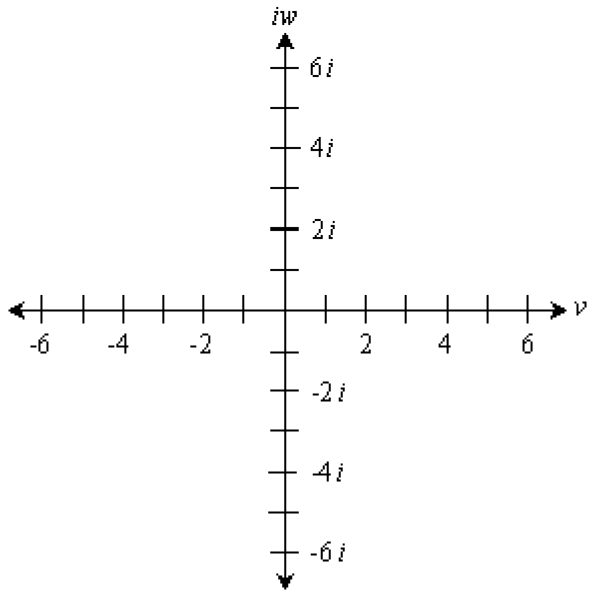
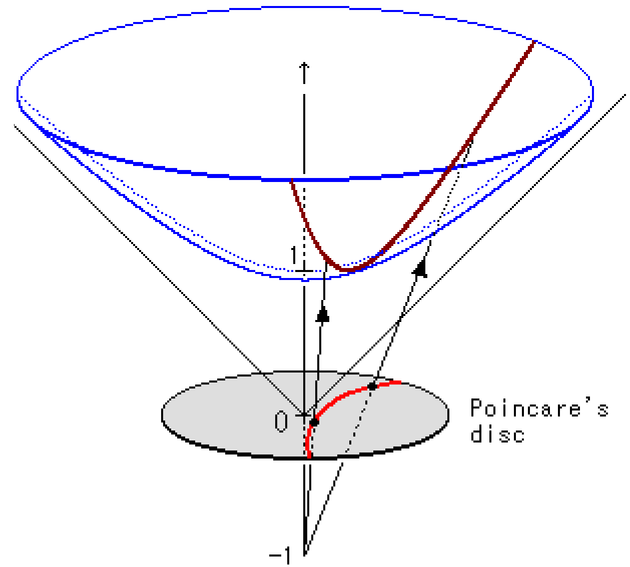
Relation to scalar curvature

Qubit state family	Coherent state family	Squeezed state family
		
Bloch sphere	Complex plane	Unit disk=Hyperboloid
Scalar curvature 2	Scalar curvature 0	Scalar curvature -4

$$w = w_0|0\rangle + w_1|1\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Thanks to http://www1.kcn.ne.jp/~iittoo/index.htm

Optimal performance with the second order asymptotics

Fidelity | **Square of Bures' dis.**

Qubit: $\frac{n+1}{n+2} \approx 1 - \frac{1}{n} + (1+1)\frac{1}{n^2}, \quad \frac{2}{2n+3} \approx \frac{1}{n} - \left(\frac{1}{2} + 1\right)\frac{1}{n^2}$

coherent: $\frac{n}{n+1} \approx 1 - \frac{1}{n} + (1+0)\frac{1}{n^2}, \quad \frac{2n}{2n+1} \approx \frac{1}{n} - \left(\frac{1}{2} + 0\right)\frac{1}{n^2}$

Squeezed: $\frac{n-2}{n-1} \approx 1 - \frac{1}{n} + (1-2)\frac{1}{n^2}, \quad \frac{2}{2n-3} \approx \frac{1}{n} - \left(\frac{1}{2} - 2\right)\frac{1}{n^2}$

Scalar curvature

Qubit state family: $2 = 2 \times 1$

Coherent state family: $0 = 2 \times 0$

Squeezed state family: $-4 = 2 \times -2$

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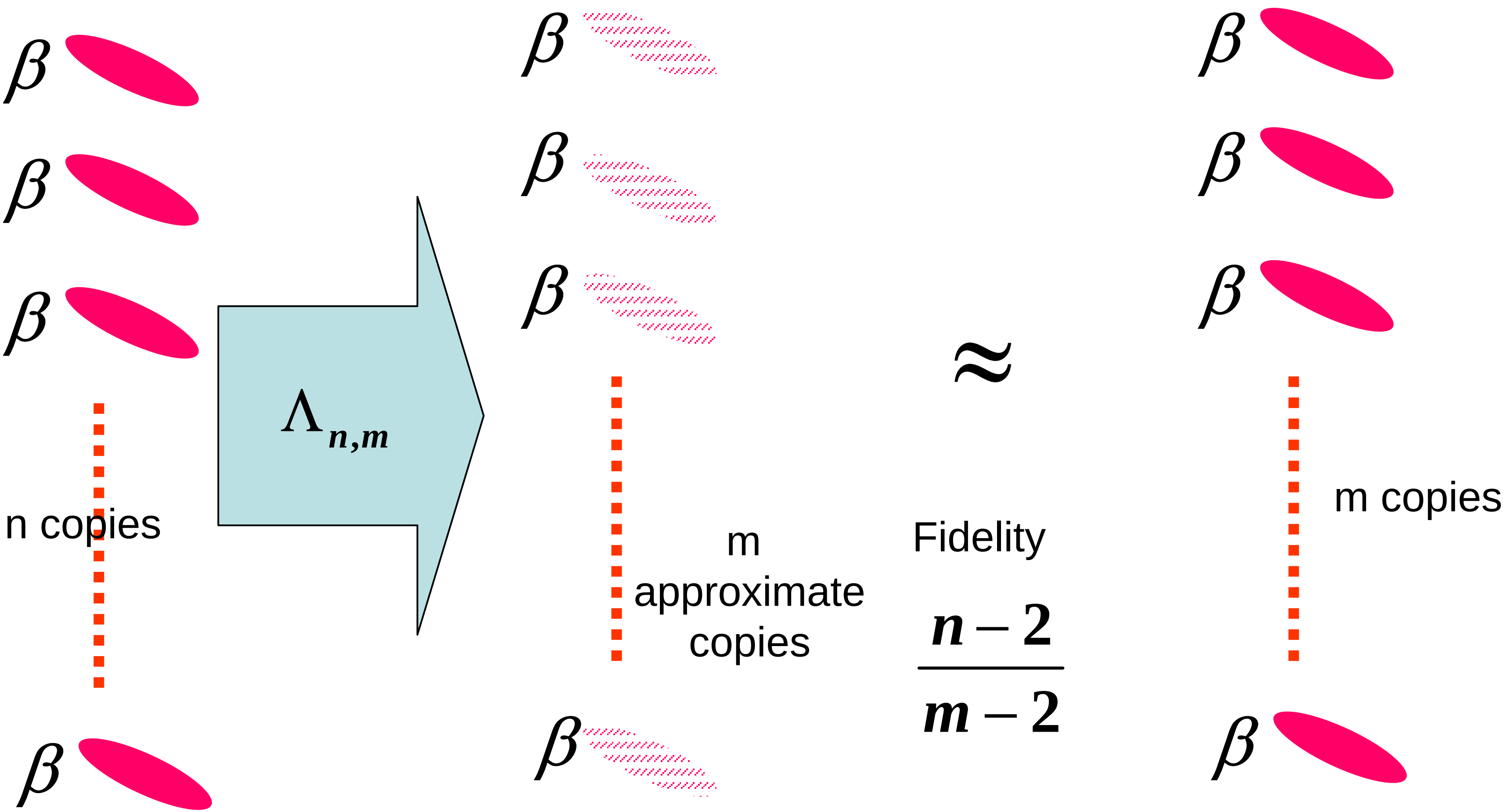
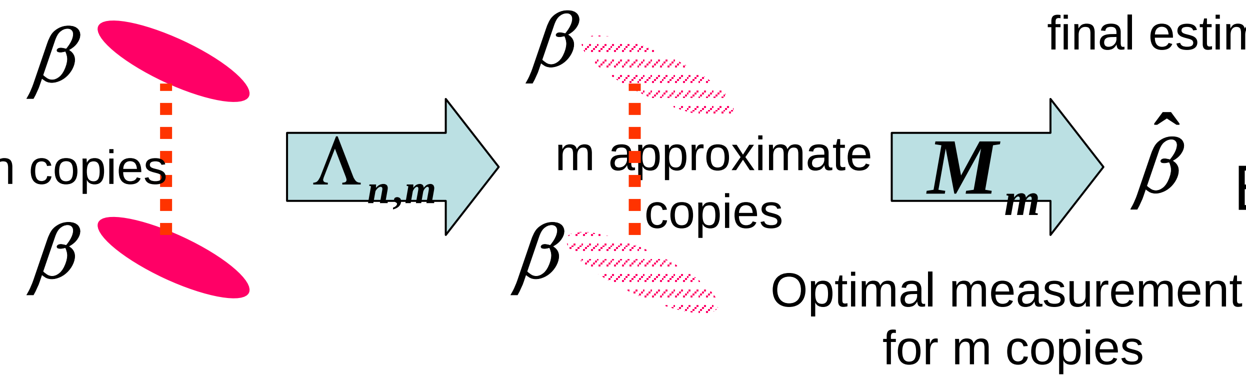
Optimal cloning of squeezed state



Optimal cloning does not lose information

Cloning + Optimal measurement



Both distributions of $\hat{\beta}$ are coincident.

Direct application of the optimal measurement

The same result can be shown in the qubit (qudit) case and the coherent state case.

Optimal cloning of squeezed state

Initial family $\left\{\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right\}$**, Target family** $\left\{\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}\right\}$**.**

Λ **: covariant, i.e.,**

$$\Lambda(V(g)^{\otimes n}\rho(V(g)^{\otimes n})^{\dagger}) = V(g)^{\otimes m}\Lambda(\rho)(V(g)^{\otimes m})^{\dagger}.$$

Optimize the fidelity ${}_s\langle\beta|^{\otimes m}\Lambda\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)|\beta\rangle_s^{\otimes m}$.

If $m \geq n > 2$, **the optimal cloning is**

$$\Lambda_{n,m}(\rho) \triangleq \frac{n-2}{m-2} P_m\left(\rho \otimes I^{\otimes(m-n)}\right) P_m.$$

$$\Lambda_{n,m}\left(\left(|0\rangle_{s\ s}\langle 0|\right)^{\otimes n}\right) = \sum_{k=0}^{\infty} \frac{(k+\frac{m-n}{2}-1)\cdots\frac{m-n}{2}(n-2)}{(k+\frac{m}{2}-1)\cdots\frac{m}{2}(m-2)} |k\rangle_{N\ N}\langle k|,$$

where $|k\rangle_m \triangleq \frac{1}{c_k}\left(\left(a^{(m)}\right)^{\dagger}\right)^k |0\rangle^{\otimes m}$.

$${}_s\langle\beta|^{\otimes m}\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)|\beta\rangle_s^{\otimes m} = \frac{n-2}{m-2}$$

Optimal cloning does not lose information

The optimal asymptotic error of the family

$$\left\{\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)\right\}$$ **equals that of family** $$\left\{\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right\}.$$

That is, if we perform the measurement $\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}$

for the family $\left\{\Lambda_{n,m}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\right)\right\}$, **the data obey**

the distribution $$\left(\frac{n}{2}-1\right)\left(1-\left|\frac{\beta-\hat{\beta}}{\beta\overline{\hat{\beta}}-1}\right|^2\right)^{\frac{n}{2}}\frac{d^2\hat{\beta}}{\pi(1-|\hat{\beta}|^2)^2},$$

which gives the distribution when we apply the optimal measurement to *n* copies.
Similar phenomenon happens in the case of coherent state.

Its reason

We focus on the dual map $\Lambda_{n,m}{}^{*}$ of $\Lambda_{n,m}$.

$$(m-2)\Lambda_{n,m}{}^{*}\left(\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}\right)\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

$$=(m-2)\frac{n-2}{m-2}\mathrm{Tr}_{H^{\otimes(m-n)}}\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes m}\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

$$=(n-2)\left(|\beta\rangle_{s\ s}\langle\beta|\right)^{\otimes n}\frac{d^2\beta}{\pi(1-|\beta|^2)}$$

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